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COMMENT

Action and pseudocharge of a free-space electromagnetic wave

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Abstract. The action and pseudocharge of a free-space electromagnetic wave are shown to be identically zero. These results are Lorentz and gauge invariant.

1. Introduction

Based on an example by Chu and Ohkawa [1], Khare and Pradhan [2] have recently constructed a free-space electromagnetic wave with non-zero action S and non-zero pseudocharge Q. Although their analysis was technically flawed and subsequently modified, Khare and Pradhan [3, 4] still maintained their assertion about the existence of such a wave. The coefficient $C(k)$ of their Fourier decomposition of the vector potential behaves singularly as $1/k^2$ near $k=0$. Herein lies the essence of their example, but as Michel [5] has indicated, this is equivalent to a superposition of uniform static electric and magnetic fields.

The question of whether there exists a free-space electromagnetic wave with non-zero S and non-zero Q has finally been answered in the negative by Brownstein [6]. His proof is developed assuming spatially bounded electric and magnetic fields and choosing a particular gauge in which the scalar potential vanishes. The purpose of this comment is to lend further support to these results by deriving them in the most general fashion without making any specific requirements on the gauge.

2. Equations of motion and invariants of the electromagnetic field [7]

The electromagnetic field can be described by the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1}$$

and its dual

$$\tilde{F}_{\mu\nu} = \varepsilon_{\mu\nu\alpha\beta} F^{\alpha\beta} \tag{2}$$

where A_μ is the 4-potential. In free space, the equations of motion are

$$\partial_\alpha F^{\alpha\beta} = 0 \tag{3}$$

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0. \tag{4}$$

We can construct two Lorentz and gauge invariants with these tensors: $F_{\mu\nu}F^{\mu\nu}$ which is a scalar and $F_{\mu\nu}\tilde{F}^{\mu\nu}$ which is a pseudoscalar.

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3. The action

The action S can be defined by

$$S=\int \mathrm{d}x\, F_{\mu\nu}F^{\mu\nu}. \tag{5}$$

To calculate the action, we need to evaluate $F_{\mu\nu}$. To do so, we Fourier decompose the 4-potential

$$A_\mu(x)=\int \mathrm{d}k \exp(\mathrm{i}kx)a_\mu(k) \tag{6}$$

where $kx=k_\mu x^\mu$, and the integration is over all four components. The Fourier decomposition of $A_\mu(x)$ assumes that $a_\mu(k)$ approaches zero fast enough as k goes to infinity so as to ensure the existence of the various derivatives needed to construct the electromagnetic field tensor. This is equivalent to the requirement of spatially bounded electric and magnetic fields which, although from a mathematical point of view limits the class of fields considered, is nevertheless necessary from the physical standpoint so that the invariants be meaningful well defined quantities. Applying (1), we obtain

$$F_{\mu\nu}=\mathrm{i}\int \mathrm{d}k \exp(\mathrm{i}kx)(k_\mu a_\nu(k)-k_\nu a_\mu(k)) \tag{7}$$

and the equation of motion (3) yields

$$k^\mu k_\mu a_\nu(k)-k^\mu k_\nu a_\mu(k)=0 \tag{8}$$

as the equation of motion for the Fourier components.

We now calculate the action (5) using (7) and obtain

$$S=-\int \mathrm{d}x\,\mathrm{d}k\,\mathrm{d}k' \exp[\mathrm{i}(k+k')x](k'_\mu a_\nu(k')-k'_\nu a_\mu(k')(k^\mu a^\nu(k)-k^\nu a^\mu(k)). \tag{9}$$

Performing the x integration, we obtain $(2\pi)^4\delta(k+k')$, which allows us to integrate over k', giving

$$S=(2\pi)^4\int \mathrm{d}k(k_\mu a_\nu(-k)-k_\nu a_\mu(-k))(k^\mu a^\nu(k)-k^\nu a^\mu(k)). \tag{10}$$

The integrand in (10) yields

$$k_\mu k^\mu a_\nu(-k)a^\nu(k)+k_\nu k^\nu a_\mu(-k)a^\mu(k)-k_\mu k^\nu a_\nu(-k)a^\mu(k)-k_\nu k^\mu a_\mu(-k)a^\nu(k) \tag{11}$$

which reduces to

$$2(k_\mu k^\mu a^\nu(k)-k_\mu k^\nu a^\mu(k))a_\nu(-k) \tag{12}$$

and is identically zero by making use of the equation of motion (8). Therefore, the action S is also identically zero.

4. The pseudocharge

The pseudocharge Q can be defined by

$$Q=\int \mathrm{d}x\, F_{\mu\nu}\tilde{F}^{\mu\nu}. \tag{13}$$

Applying (2) and (7), we obtain

$$\tilde{F}_{\mu\nu} = \mathrm{i}\int \mathrm{d}k\, \exp(\mathrm{i}kx)\varepsilon_{\mu\nu\alpha\beta}(k^\alpha a^\beta(k) - k^\beta a^\alpha(k)). \tag{14}$$

We now calculate the pseudocharge (13) by applying (7) and (14) together:

$$Q = -\int \mathrm{d}x\, \mathrm{d}k\, \mathrm{d}k'\, \exp[\mathrm{i}(k+k')x]\varepsilon_{\mu\nu\alpha\beta}(k^\alpha a^\beta(k) - k^\beta a^\alpha(k))(k'^\mu a^\nu(k') - k'^\nu a^\mu(k')). \tag{15}$$

We proceed as in (9) performing the x integral to obtain $(2\pi)^4\delta(k+k')$ followed by integration over k' to yield

$$Q = (2\pi)^4 \int \mathrm{d}k\, \varepsilon_{\mu\nu\alpha\beta}(k^\alpha a^\beta(k) - k^\beta a^\alpha(k))(k^\mu a^\nu(-k) - k^\nu a^\mu(-k)). \tag{16}$$

The integrand in (16) is made up of factors of the form $\varepsilon_{\mu\nu\alpha\beta}k^\alpha k^\mu$, each of which vanishes by virtue of $\varepsilon_{\mu\nu\alpha\beta} = -\varepsilon_{\nu\mu\alpha\beta}$ resulting from the antisymmetric character of this unit fourth-rank tensor and therefore the pseudocharge Q is also identically zero.

5. Conclusions

Landau and Lifshitz [7] have shown that, if we consider the complex vector $\boldsymbol{F} = \boldsymbol{E} + \mathrm{i}\boldsymbol{H}$, the only invariant under Lorentz transformations (which are simply rotations in 4-space) that we can construct is its square: $\boldsymbol{F}^2 = (E^2 - H^2) + 2\mathrm{i}\boldsymbol{E}\cdot\boldsymbol{H}$. Thus, the real quantities $(E^2 - H^2)$ and $\boldsymbol{E}\cdot\boldsymbol{H}$ (which up to a numerical factor are $F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}\tilde{F}^{\mu\nu}$ respectively) are the only two independent invariants of the tensor $F_{\mu\nu}$. The use of the duality transformation in the second part of Brownstein's proof [6] is justified by this particular fact.

We have therefore shown that the action S and the pseudocharge Q of an electromagnetic wave in free space are both identically zero. These results are Lorentz and gauge invariant.

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